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1989 J. Phys. A: Math. Gen. 22 L861

(http://iopscience.iop.org/0305-4470/22/18/001)

J. Phys. A: Math. Gen. **22** (1989) L861–L863. Printed in the UK

LETTER TO THE EDITOR

Triviality of the prolongation algebra of the Kuramoto–Sivashinsky equation

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Received 25 April 1989

Abstract. We apply the well known Wahlquist–Estabrook prolongation technique to the Kuramoto–Sivashinsky equation. The prolongation algebra turns out to be trivial in the sense that it is commutative. This supports non-integrability of the equation.

The Kuramoto–Sivashinsky equation is a non-linear partial differential equation frequently encountered in the study of continuous media. It describes, for instance, the fluctuation of a flame front or an oscillating chemical reaction in a homogeneous medium. A review may be found in [1]. In conservative form it is written as

$$u_t + uu_x + \mu u_{xx} + \nu u_{xxxx} = 0. \tag{1}$$

This equation is invariant under the Galilean transformation

$$(u, x, t) \rightarrow (u + c, x - ct, t). \tag{2}$$

In [1] the well known Painlevé analysis is applied to the Kuramoto–Sivashinsky equation (1), indicating non-integrability of this equation. This is not surprising because it is known that the Kuramoto–Sivashinsky equation exhibits chaotic behaviour. In this letter we prove, following the well known prolongation technique of Wahlquist and Estabrook, that the prolongation algebra for the Kuramoto–Sivashinsky equation is trivial in the sense that it is commutative. This result is also a strong indication of the non-integrability of the equation. Although there is no rigorous proof that commutativity of the prolongation algebra implies non-integrability, it is a general conjecture that this is indeed the case. This conjecture is supported by many known examples, e.g. the Korteweg–de Vries–Burgers equation [2].

In order to write equation (1) as a system of differential forms we introduce $p = u_x$, $q = u_{xx}$, $r = u_{xxx}$. Equation (1) can now be written as

$$u_t + up + \mu q + \nu r_x = 0. \tag{3}$$

In order not to discuss a degenerate case we assume $\nu \neq 0$. When we adopt in the six-dimensional space of dependent and independent variables $\{x, t, u, p, q, r\}$ the basis forms $\{dx, dt, du, dp, dq, dr\}$, we can represent the equation (3) by the following set of 2-forms:

$$\begin{aligned} \alpha_1 &= du \wedge dt - p\, dx \wedge dt \\ \alpha_2 &= dp \wedge dt - q\, dx \wedge dt \\ \alpha_3 &= dq \wedge dt - r\, dx \wedge dt \\ \alpha_4 &= \nu\, dr \wedge dt - (\mu q + up)\, dx \wedge dt - du \wedge dx. \end{aligned} \tag{4}$$

0305-4470/89/180861+03$02.50

Let $S=\{(x, t, u(x, t), p(x, t), q(x, t), r(x, t))\}$ be a regular two-dimensional manifold in six-dimensional space. Any S that is a solution of (3) will annul the set of forms (4) and vice versa [3]. The ideal of 2-forms (4) is closed (i.e. $\Sigma_{i=1}^{4} \eta_i \wedge \alpha_i$, $i=1, \ldots, 4$, η_i being a set of 1-forms) which means that equation (3) satisfies the integrability conditions. To obtain a prolongation structure we introduce the prolongation variable y (the number of prolongation variables will not influence the process and will therefore be taken equal to 1). We also introduce the 1-form $\omega=\mathrm{d}y+F\,\mathrm{d}x+G\,\mathrm{d}t$. Following [3] we take $F=F(u, p, q, r, y)$ and $G=G(u, p, q, r, y)$. The extended ideal $I(\alpha_1, \ldots, \alpha_4, \omega)$ should be closed now, which means in particular that $\mathrm{d}\omega=\Sigma_{i=1}^{4} f_i\alpha_i+n \wedge \omega$ (f_i are 0-forms and n is a 1-form). We define $[G, F]=GF_y-G_yF$ and obtain the following equations for F and G:

$$F_p=0 \qquad F_q=0 \qquad F_r=0 \tag{5}$$

$$\nu F_u+G_r=0 \tag{6}$$

$$[G, F]+pG_u+qG_p+rG_q-G_r(\mu q+up)/\nu=0. \tag{7}$$

Equations (5) give $F=F(u, y)$ and integrating (6) renders

$$G=-\nu rF_u+G^1(u, p, q, y).$$

Substituting this in equation (7) we derive an equation linear in r. After separating powers of r and integration with respect to q this yields

$$G^1=\nu q(pF_{uu}+[F_u, F])+G^2(u, p, y)$$

which gives, after substituting this in the remaining part, an equation quadratic in q. Collecting powers gives the following three equations:

$$F_{uu}=0 \tag{8}$$

$$2\nu p[F_{uu}, f]+\nu[[F_u, F], F]+\nu p^2F_{uuu}+\mu F_u+G_p^2=0 \tag{9}$$

$$[G^2, F]+pG_u^2+upF_u=0. \tag{10}$$

From equation (8) we find $F=X_1+uX_2$. Let us introduce the shorthand notation $[X_i, X_j]=[i, j]$ and JI(i, j, k) for 'Jacobi identity applied for X_i, X_j and X_k'. Further we define

$$[1,2]=X_4 \qquad [1,4]=X_5 \qquad [1,5]=X_6 \qquad [1,6]=X_7. \tag{11}$$

The result for F together with relations (2.9) gives us, after integrating equation (9) with respect to p,

$$G^2=-p(\mu X_2+\nu X_5+\nu u[2,4])+G^3(u, y).$$

Together with (10) this gives an equation quadratic in p and after collecting powers we find

$$[2,4]=0 \tag{12}$$

$$-[(\mu X_2+\nu X_5+\nu u[2,4]), (X_1+uX_2)]+uX_2+G_u^3=0 \tag{13}$$

$$[G^3, (X_1+uX_2)]=0. \tag{14}$$

JI$(1,2,4)$ together with (12) gives $[2,5]=0$. With this we find after integrating (13)

$$G^3=-\tfrac{1}{2}u^2X_2-u(\mu X_4+\nu X_6)+X_3.$$

Together with (14) this leads to a quadratic equation in u which splits up into

$$[1,3]=0 \qquad [2,3]=\mu X_5+\nu X_7 \qquad [2,6]=-X_4/2\nu \qquad [2,4]=0 \tag{15}$$

as we already found. Relations (15) together with the relations for $X_1, \ldots, X_4$ (11) define the prolongation algebra.

Theorem. The prolongation algebra of the Kuramoto-Sivashinsky equation determined by (11) and (15) is commutative.

Proof. By use of the Jacobi identity we shall prove $\mathrm{JI}(2,5,6) = 3X_4/4\nu^2$. We already found by use of $\mathrm{JI}(1,2,4)$ that $[2,5]=0$. This relation together with $\mathrm{JI}(1,2,5)$ leads to

$$[4,5] = X_4/2\nu. \tag{16}$$

From $\mathrm{JI}(1,2,3)$ we find a relation between two commutators

$$[3,4] = -\mu X_6 - \nu[1,7].$$

Using (16), $\mathrm{JI}(1,4,5)$ gives $[4,6] = X_5/2\nu$. From $\mathrm{JI}(1,2,6)$ we find $[2,7] = -X_5/\nu$. Now $\mathrm{JI}(1,2,7)$ gives

$$[2,[1,7]] + [4,7] + X_6/\nu = 0 \tag{17}$$

and from $\mathrm{JI}(2,3,4)$ we have

$$-\nu[2,[1,7]] + \nu[4,7] + \mu X_4/\nu = 0. \tag{18}$$

When we take an appropriate linear combination of (17) and (18) we can eliminate $[2,[1,7]]$ and therefore we have

$$[4,7] = -\mu X_4/2\nu^2 - X_6/2\nu.$$

This result we use to calculate $\mathrm{JI}(1,4,6)$ which gives us

$$[5,6] = \mu X_4/2\nu^2 + X_6/\nu.$$

This result we use to show that

$$\mathrm{JI}(2,4,6) = 3X_4/4\nu^2$$

To fulfil this last Jacoby identity X_4 has to be zero. It follows easily that $X_5=0$, $X_6=0$, $X_7=0$ and therefore $[1,2]=0$ and $[2,3]=0$. We already had $[1,3]=0$, which proves the theorem.

Remark. Adding an additional term of the form δu_{xxx} to the orginal equation (1) yields the equation

$$u_t + uu_x + \mu u_{xx} + \delta u_{xxx} + \nu u_{xxxx} = 0. \tag{19}$$

In the same way as previously, we can show that the prolongation algebra commutes in this case also. So for equation (19) also no non-trivial prolongation algebra exists.

In conclusion, we have shown for $\nu \neq 0$ that (11) and (15) define the prolongation algebra for the Kuramoto-Sivashinsky equation. This implies for the generators X_1, X_2 and X_3 that they commute. Therefore no non-trivial prolongation algebra exists for the Kuramoto-Sivashinsky equation.

References

[1] Conte R and Mussette M 1989 *J. Phys. A: Math. Gen.* **22** 169
[2] Feudel F and Steudel H 1985 *Phys. Lett.* **107A** 5
[3] Wahlquist H D and Estabrook F B 1975 *J. Math. Phys.* **16** 1